... writing down the balance of forces for link i

$$f_i - {}^{i}_{i+1}R f_{i+1} + m_i g_i = m_i a_{ci} \quad (2)$$

Note: This is a ~~pictorial~~ vectorial calculation.

balance of tourque for link i.

$$\tau_i - {}^{i}_{i+1}R \tau_{i+1} + f_i \times r_{i,ci} - {}^{i}_{i+1}R f_{i+1}$$

$$= I_i \alpha_i + \omega_i \times (I_i \omega_i) \quad (3)$$

Newton Euler Procedure

Forward Recursion

1. Start with initial conditions

$\omega_0 = 0$ 1.

$\alpha_0 = 0$ 2.

$a_{c0} = 0$ 3.

$a_{e0} = 0$ 4.

Solve in that order

a. $\omega_i = {}^{i+1}_{i}R^T \omega_i + \hat{b}_i \dot{q}_i$

$\hat{b}_i = {}^{0}_{i}R^T \hat{z}_{i-1}$

b. $\alpha_i = {}^{i-1}_{i}R^T \alpha_{i-1} + \hat{b}_i \ddot{q}_i + \omega_i \times \hat{b}_i \dot{q}_i$

c. $a_{e,i} = {}^{i-1}_{i}R^T a_{e,i-1} + \dot{\omega}_i r_{i,i}$

$+ \omega_i \times (\omega_i \times r_{i,i+1})$

d. $a_{ci} = {}^{i-1}_{i}R^T a_{e,i-1} + \dot{\omega}_i \times r_{ici}$

$+ \omega_i \times (\omega_i \times r_{ici})$

$i = i+1$ then go to step a again until of course $i = n$.

Backward Recursion

Start w initial conditions

$f_{n+1} = 0$ & $\tau_{n+1} = 0$

let $i = n$

a. $f_i = {}^{i}_{i+1}R f_{i+1} + m_i a_{c,i} - m_i g_i$

b. $\tau_i = {}^{i}_{i+1}R \tau_{i+1} - f_i \times r_{ici}$

$+ ({}^{i}_{i+1}R f_{i+1}) \times r_{ici} + I_i \alpha_i$

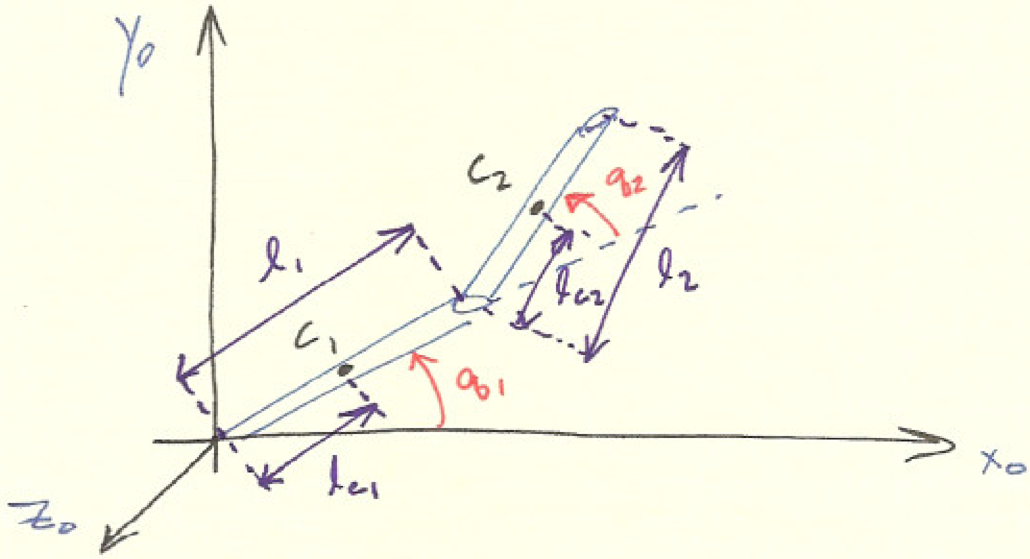
$+ \omega_i \times (I_i \omega_i)$

$i = i - 1$

if $i \geq 1$ then go to (a)

Remark: f_i $i = 1 \ldots n$ are internal forces, they do not shown up in the internal forces of motion. the τ_i are the external tourques.

EX: Planar manipulator, 2 DOF.



link 1 m_1 I_1

link 2 m_2 I_2

We begin with the forward recursion to express the various velocities and accelerations in terms of q_1 & q_2 and their derivations

$$\omega_1 = \dot{q}_1 \hat{k} \qquad \alpha_1 = \ddot{q}_1 \hat{k}$$

$$\omega_2 = (\dot{q}_1 + \dot{q}_2)\hat{k} \qquad \alpha_2 = (\ddot{q}_1 + \ddot{q}_2)\hat{k}$$

$\hat{k}$ is $\perp$ to page.

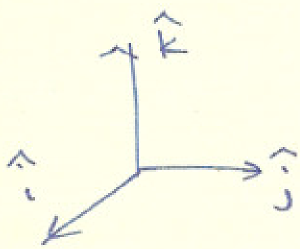
$r_{1c_1} = l_{c1}\hat{i} \quad r_{2c_1} = (l_1 - l_{c1})\hat{i} \quad r_{12} = l_1\hat{i}$

$r_{2c_2} = l_{c2}\hat{i} \quad r_{3c_2} = (l_2 - l_{c2})\hat{i} \quad r_{23} = l_2\hat{i}$

Now performing the forward recursion
link 1

$$i = 1 \quad ; \quad a_{e,0} = 0$$

$$a_{c,1} = \ddot{q}_1 \hat{k} \times l_{c1}\hat{i} + \dot{q}_1 \hat{k} \times (\dot{q}_1 \hat{k} \times l_{c1}\hat{i})$$



$\hat{k} \times \hat{i} = \hat{j}$

$\hat{k} \times \hat{j} = -\hat{i}$

$$a_{c1} = l_{c1}\ddot{q}_1\hat{j} - l_{c1}\dot{q}_1^2\hat{i} = \begin{bmatrix} -l_{c1}\dot{q}_1^2 \\ l_{c1}\ddot{q}_1 \\ 0 \end{bmatrix}$$

$$g_1 = -{}^{0}_{1}R\, g\,\hat{j} = g\begin{bmatrix} -\sin q_1 \\ -\cos q_1 \\ 0 \end{bmatrix}$$

$${}^{0}_{1}R = \begin{bmatrix} C_{q_1} & -S_{q_1} & 0 \\ S_{q_1} & C_{q_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_{e,1} = \dot{\omega}_1 \times r_{12} + \omega_1 \times (\omega_1 \times r_{12})$$

$$= \ddot{q}_1 \hat{k} \times l_1 \hat{i} + \dot{q}_1 \hat{k} \times (\dot{q}_1 \hat{k} \times l_1 \hat{i})$$

$$= l_1 \ddot{q}_1 \hat{j} + \dot{q}_1 \hat{k} \times (l_1 \dot{q}_1 \hat{j})$$

$$= l_1 \ddot{q}_1 \hat{j} - l_1 \dot{q}_1^2 \hat{i} = \begin{bmatrix} -l_1 \dot{q}_1^2 \\ l_1 \ddot{q}_1 \\ 0 \end{bmatrix}$$

Now for link 2.

$$a_{c,2} = {}^1_2R^T a_{e,1} + (\ddot{q}_1 + \ddot{q}_2)\hat{k} \times l_{c2}\hat{i}$$

$$+ (\dot{q}_1 + \dot{q}_2)\hat{k} \times \left[(\dot{q}_1 + \dot{q}_2)\hat{k} \times l_{c2}\hat{i}\right]$$

$${}^1_2R^T a_{e,1} = \begin{bmatrix} C_{q2} & S_{q2} & 0 \\ -S_{q2} & C_{q2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1 \dot{q}_1^2 \\ l_1 \ddot{q}_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 \dot{q}_1^2 C_{q2} + l_1 \ddot{q}_1 S_{q2} \\ l_1 \dot{q}_1^2 S_{q2} + l_1 \ddot{q}_1 C_{q2} \\ 0 \end{bmatrix}$$

$$a_{c,2} = \begin{bmatrix} -l_1 \dot{q}_1^2 C_{q2} + l_1 \ddot{q}_1 S_{q2} - l_{c2}(\dot{q}_1 + \dot{q}_2)^2 \\ l_1 \dot{q}_1^2 S_{q2} + l_1 \ddot{q}_1 C_{q2} + l_{c2}(\ddot{q}_1 + \ddot{q}_2)^2 \\ 0 \end{bmatrix}$$

The gravity vector g is given by

$$g_2 = g\begin{bmatrix} \sin(q_1 + q_2) \\ -\cos(q_1 + q_2) \\ 0 \end{bmatrix}$$

? are you sure this is ~~write~~.

Allen would like to point out that it is "right" not "write"!

$$= -{}^{0}_{2}R^{T}\begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

Now looking at the backward recursion method.

Starting with link 2.

$i = 2$

$f_3 = 0$

$\tau_3 = 0$

$f_2 = m_2 a_{c,2} - m_2 g_2$

$\tau_2 = I_2 \alpha_2 + \omega_2 \times (I_2 \omega_2) \cancel{-} f_2 \times l_{c2}\hat{i}$

$\omega_2 \times (I_2 \omega_2) = 0$

Since ω_2 and $I_2\omega_2$ are both aligned with $\hat{k}$ (rotation just about z)

$$\tau_2 = I_2(\ddot{q}_1 + \ddot{q}_2)\hat{k} + \Big[m l_1 l_{c2}^{2}(\ddot{q}_1 + q_2) + m_2 l_{c2}\, g\cos(q_1 + q_2)\Big]$$

link 1

$$i = 1$$

$$f_1 = m_1 a_{c,i} + {}^1_2R f_2 - m_1 g_1$$

$$\tau_1 = {}^1_2R \tau_2 - f_1 \times l_{c1} \hat{i} - ({}^1_2R f_2) \times (l_1 - l_{c1}) \hat{i}$$

$$+ I_1 \alpha_1 + \underbrace{\omega_1 \times (I_1 \omega_1)}_{=0}$$

$${}^1_2R \tau_2 = \tau_2$$

Since the rotation matrix does not affect the third component of vectors. and

$$\tau_2 = \hat{k} \tau_2$$

$$\omega_1 \times (I_1 \omega_1) = 0$$

Since ω_1 and $I_1 \omega_1$ are aligned in $\hat{z}$

$$\tau_1 = \tau_2 - m_1 a_{c,1} \times l_{c1} \hat{i} + m g_1 \times l_{c1} \hat{i}$$

$$- ({}^1_2R f_2) \times l_1 \hat{i} + I_1 \alpha_i$$